Horizon Mechanics and Asymptotic Symmetries with a Immirzi-like Parameter in 2+1 Dimensions

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Starting with a generalized theory of $2+1$ gravity containing an Immirzi like parameter, we derive the modified laws of black hole mechanics using the formalism of weak isolated horizons. Definitions of horizon mass and angular momentum emerge naturally in this framework. We further go on to analyze the asymptotic symmetries, as first discussed by Brown and Henneaux, and analyze their implications in a completely covariant phase space framework.

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I. INTRODUCTION

Gravity in $2+1$ dimensions [1, 2] has been an active research arena primarily because it is an example of exactly solvable quantum system [3]. The aim of this program also lies in gaining knowledge about quantum gravity phenomena for the difficult problem in $3+1$ dimensions (reference [4] has detail discussions). Though the theory is fairly understood in certain topological set up [5, 6] (*i.e.*, when the spatial foliations are compact Riemann surfaces), not everything is trivial when there is an inner boundary. Perhaps the most interesting, and hence most studied of these theories of $2+1$ gravity is the one with a negative cosmological constant which has been shown to admit the BTZ black hole as an excited state and the AdS_3 solution as it's vacuum [7]. Although there are a more general class of black holes in 2+1 topological gravity theories [8], of which BTZ is a special one. In the first order formalism, this theory is equivalent to $SO(2,1) \times SO(2,1)$ Chern-Simons gauge theory. Chern-Simons is again a purely topological theory. The local lorentz transformation and the diffeomorphisms are only local excitations, which being gauge, leave the theory devoid of any local physics. Global or topological degrees of freedom are the only ones to look for while constructing the physical dynamics of gravity in 2+1 dimensions. This is in contrast to the much studied topologically massive gravity (TMG) theory [9] (or it's recently understood ramifications [10, 11]) in which one introduces local degrees of freedom, a parity violating massive graviton. It contains the usual Einstein-Hilbert term, the gravitational Chern-Simons *and* a cosmological constant. It has also been suggested that a three dimensional gravity can always be transformed to TMG gravity through field redefinition and a consistent truncation [12]. The theory of TMG has some peculiarities - the massive excitations carry negative energy for a positive coupling constant (in this case, it is the G) [9]. In case of negative cosmological constant TMG, the situation is drastic. Change in sign of the coupling constant gives excitations with positive energy but gives negative mass BTZ black hole solutions [13].

Black holes in such theories (see [13–16] and references therein) and their entropy have been studied in great detail [17, 18] and in [19] exhaustively for a large class of interactions governed by Chen Simons theory. In three dimensions, black hole entropy calculations are majorly based upon two different routes. Most popular is the one which follows [20]; this again is based on the results of the seminal paper by Brown and Henneaux [21]. They showed that asymptotic symmetries of a solution of $2+1$ general relativity with negative cosmological constant (not necessarily the BTZ solution), is given by a pair of Witt algebras- the deformation algebra of S^1 instead of the expected isometry $SO(2,2)$ of AdS_3. Canonical phase space realization of these asymptotic symmetries however are given by a pair of Virasoro algebras, which are centrally extended version of the symmetry algebra. A simple use of Cardy formula for the central extensions gives the entropy (see [22] for discussions). On the other hand, there is another path, (eg [17, 19]) which uses covariant phase space framework, following Wald [42].

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Unfortunately this approach heavily relies on the (bifurcate) Killing horizon structures, which have their own problems including restriction to non-extremal horizons only. Dynamical issues, conserved charges in similar class of theories, including the canonical realization of asymptotic symmetries have been studied in [18, 23–28]. Entropy of the BTZ black hole in these modified topological theories and the TMG were also presented in these papers. Contrary to the Bekenstein-Hawking expectation, the entropy turns out not only to be proportional to the black hole area, but also to some extra terms, involving even the horizon angular momentum [18, 28]. In this paper, we shall investigate related issues for a general class of theories in a covariant manner and show that such results are expected. Moreover, we shall establish that our method is equally applicable to extremal and non-extremal black holes since it does not rely on the existence of bifurcation spheres.

Generalized versions of $2+1$ topological gravity which retains its topological nature came into prominence through [29]. In the present work, we consider a special case of such generalized theory [30, 31]. More precisely, we shall work with a theory having a negative cosmological constant and a parameter which imitates the Barbero-Immirzi parameter of $3+1$ gravity [32]. The possibility of this generalization also was hinted in the pioneering work [3]. The gauge group still remains $SO(2,1) \times SO(2,1)$. This theory is interestingly linked to TMG in a subtle way, as discussed in [6]. When one forces the torsion (T^I) to be zero, *i.e.* goes to the partial solution space of $T^I = 0$, one lands on TMG. The limit $\gamma \to 1$, of the new parameter (which behaves as chiral parameter in TMG literature), has interesting consequences both in the purely topological and the massive theories [6, 27, 33–36].

Our aim will be to establish the laws of black hole mechanics in this theory and to determine entropy in fully covariant framework, and on doing so we will show how our approach fills up the gaps in existing literature. In this respect, it becomes important to introduce conserved charges like the angular momentum and mass. We shall use the formalism of isolated horizons to address these issues. The set-up of isolated horizons is robust and conceptually straightforward, resulting in surprising simplicity in calculations. The details of isolated horizon formalism were developed in a series of papers [37–40]. The isolated horizon formalism for general relativity in $2+1$ dimensions was developed in [41]. We shall however use a weaker set of boundary conditions than [41], extend to more general theories, study the asymptotic symmetries and eventually determine the entropy of horizons.

The basic idea is the following: a horizon (black hole or cosmological) is a null hypersurface which can be described locally, by providing the geometric description of that surface only. The black hole horizon (we are interested in these horizons here) is described in this formalism to be an internal boundary of spacetime which is expansion free and on which the field equations hold[1]. Unlike the Killing horizons/event horizons, we need not look *outside/asymptotic* or the space time in the vicinity of horizon to define isolated horizons; only horizon properties are enough. It is because of this generality that isolated horizon is useful to describe even solutions where the asymptopia is still not well-defined or has not developed yet. More precisely, all event horizons/Killing horizons are isolated horizons but not all isolated horizons are Killing/event horizon. As it happens (and we shall show this below), the boundary conditions enable us to prove the zeroth law of black hole mechanics directly. The first law of black hole mechanics and construction of conserved charges is not difficult in this formalism. Given a field theory, there exists a straightforward way which enable us to covariantly construct the space of solutions (and a symplectic structure on the covariant phase-space), initially introduced in [46]. This has been applied successfully to study dynamics of space times with isolated horizon as an internal boundary. The conserved charges (like angular momentum) are precisely the *Hamiltonian functions* corresponding to the vector field generating canonical transformations or the so called *Hamiltonian vector fields* (which in this case is related to rotational Killing vector field on). The first law is, in this description, the necessary and sufficient condition for the null generator of the horizon to be a Hamiltonian vector field. These features, as we shall show below, can be established very easily.

There is a precise definition of the asymptotic symmetry group, if we know the fall of behaviour of the geometry asymptotically. A natural question to ask is whether the action of this group on the pre-symplectic manifold, defined through the degenerate symplectic structure, is a Hamiltonian. As have been the expectation through canonical analyses made earlier, the answer is not in the affirmative, rather the algebra of symmetry genetrators get centrally extended. This leads to finding the black hole entropy as discussed earlier (just as for the TMG case). This is similar to TMG where the parameters are the topological mass and the cosmological constant. (For TMG, this implies that the massive graviton introduced through the extra couplings have no effect on the entropy.)

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[1] This construction is more general than that of Killing horizon. Laws of black hole mechanics were proved for this quasi-local definition too [42, 43]. But, as mentioned earlier this formalism does not seem useful to address extremal horizons. Improvements by introducing extremal horizons in the same space of the non-extremal horizons in the isolated horizons framework were made in [44, 45].

Plan of the paper is as following: In the section (II), we shall recall the definition of Weak Isolated Horizons (WIH) and prove the zeroth law of black hole mechanics [44, 45]. The proof of zeroth law is purely kinematical and does not require any dynamical information. In section (III A), we shall first discuss the generalised theories in $2+1$ dimensions and then introduce the theory with γ-parameter and negative cosmological constant. This section will also include a discussion of the BTZ solution as an example of a black hole solution in this theory. Since we shall be interested in manifolds with inner and outer boundaries we need to establish that the variational principle is well defined. In sections (III B) and (III C), we shall establish that indeed the action principle is well defined even when the inner boundary is a WIH. In section (III D), we construct the space of solutions and symplectic structure. The phase-space contains all solutions, (extremal as well as non-extremal black hole solutions) which satisfy the boundary conditions of WIH for the inner boundary and are asymptotically AdS at infinity. In section (III F), we shall show how the angular momentum can be extracted from the symplectic structure. The angular momentum will naturally arise as a Hamiltonian function (on the phase-space) corresponding to the Hamiltonian vector field associated with rotational Killing vector field on the spacetime. When the definition is applied to the BTZ solution, it will naturally arise that the angular momentum depends on the parameters J and M of the solution. In section (IV), we shall construct the vector fields which generate diffeomorphisms preserving the asymptotic conditions. We shall construct Hamiltonians functions corresponding to these vector fields and show that in presence of a WIH inner boundary, the Hamiltonian charges do not realize the algebra of vector fields. The difference is a central extension which gives rise to the entropy for black holes in these theories. We shall also observe that the parameter γ shows up in all stages. We shall discuss these issues in the section (V).

II. WEAK ISOLATED HORIZON: KINEMATICS

We now give a very brief introduction to weak isolated horizons [44]. Let $\mathcal{M}$ be a three-manifold equipped with a metric g_{ab} of signature $(-,+,+)$. Consider a null hypersurface Δ in $\mathcal{M}$ of which ℓ^a is a future directed null normal. However, if ℓ^a is a future directed null normal, so is $\xi\ell^a$, where ξ is any arbitrary positive function on Δ. Thus, Δ naturally admits an equivalence class of null normals $[\,\xi\ell^a\,]$. The hypersurface Δ being null, the metric induced on it by the spacetime metric g_{ab} will be degenerate. We shall denote this degenerate metric by $q_{ab} \triangleq g_{\underleftarrow{ab}}$ (since we are using abstract indices, we shall distinguish intrinsic indices on Δ by pullback and $\triangleq$ will mean that the equality holds *only on* Δ). The *inverse* of q_{ab} will be denoted by q^{ab} such that $q^{ab}q_{ac}q_{bd} \triangleq q_{cd}$. The expansion $\theta_{(\ell)}$ of the null normal ℓ^a is then defined by $\theta_{(\ell)} = q^{ab}\nabla_a\ell_b$, where ∇_a is the covariant derivative compatible with g_{ab}. Null surfaces are naturally equipped with many nice properties. Firstly, the null normal is hypersurface orthogonal and hence is twist-free. Secondly, the ℓ^a is also tangent to the surface. It is tangent to the geodesics generating Δ. Thus, any ℓ^a in the class $[\xi\ell^a]$ satisfies the geodesic equation:

$$\ell^a\nabla_{\underleftarrow{a}}\,\ell^b \triangleq \kappa_{(\ell)}\,\ell^b. \tag{2.1}$$

We shall interpret the acceleration $\kappa_{(\ell)}$ as the surface gravity. If the null normal to Δ is such that κ vanishes, we shall call it to be extremal surface. Otherwise, the surface will be called non-extremal. The variation of κ in the null class $[\xi\ell]$ being as $\kappa_{(\xi\ell)} = \xi\kappa_{(\ell)} + \pounds_\ell\xi$.

In what follows, we shall use the Newmann-Penrose (NP) basis for our calculations. In three dimensions, this will consist of two null vectors ℓ^a and n^a and, one spacelike vector m^a They satisfy the condition $\ell.n = -1 = -m.m$ while other scalar products vanish. This basis is particularly useful for our set-up because the normal to Δ, denoted by ℓ^a can be chosen to be the ℓ^a of NP basis. The spacelike m^a will be taken to be tangent to Δ. In this basis, the spacetime metric will be given by $g_{ab} = -2\,\ell_{(a}n_{b)} + 2\,m_{(a}m_{b)}$ whereas the pullback metric q_{ab} will be simply, $q_{ab} \triangleq m_am_b$.

A. Weak Isolated Horizon and the Zeroth Law

The null surface Δ introduced above is an arbitrary null surface equipped with an equivalence class of null normals $[\xi\ell^a]$. The conditions on Δ are too general to make it resemble a black hole horizon. To enrich Δ with useful and interesting information, we need to impose some additional structures (the imposed conditions will be weaker than that in [41] in the sense that our equivalence class of null normals will be related by functions on Δ rather than constants). As we shall see, the zeroth law and the first law of black hole mechanics will naturally follow from these conditions. These definitions will be local and only provides a construction of black hole horizon and do not define a black hole spacetime which is a global object. However, if there is a global solution, like the BTZ one, then these conditions will be satisfied.

The null surface Δ, equipped with an equivalence class of null normals $[\xi\ell^a]$, will be called a *weak isolated horizon* (WIH) if the following conditions hold:

1. Δ is topologically $S^1 \times \mathbb{R}$.
2. The expansion $\theta_{(\xi\ell)} \triangleq 0$ for any $\xi\ell^a$ in the equivalence class.
3. The equations of motion and energy conditions hold on the surface Δ and the vector field $-T^a_b\xi\ell^b$ is future directed and causal.
4. There exists a 1-form $\omega^{(\xi\ell)}$ such that it is lie-dragged along the horizon Δ,

$$\pounds_{\xi\ell}\,\omega^{(\xi\ell)} \triangleq 0 \tag{2.2}$$

In the literature, Δ is called a *non-expanding horizon* (NEH) if it satisfies only the first three conditions. It is clear that that the boundary conditions for a NEH hold good for the entire class of null normals $[\xi\ell^a]$ if it is valid for one null normal in that class. The Raychaudhuri equation imply that NEHs are also shear free. Thus, NEHs are twist-free, expansion-free and shear-free and this implies that the covariant derivative of ℓ^a on Δ much simple. There exists a one form $\omega^{(\ell)}$ (see appendix VI for a Newman-Penrose type discussion), such that

$$\nabla_{\underleftarrow{a}}\ell^b \triangleq \omega^{(\ell)}_a\ell^b \tag{2.3}$$

The one form $\omega^{(\ell)}_a$ varies in the equivalence class $[\xi\ell^a]$ as

$$\omega^{(\xi\ell)} \triangleq \omega^{(\ell)} + d\ln\,\xi \tag{2.4}$$

A few other conclusions also follow. Firstly, from equations (2.1) and (2.3), it follows that $\kappa_{(\xi\ell)} \triangleq \xi\ell.\omega^{(\xi\ell)}$. Secondly, that the null normals in the equivalence class are Killing vectors on NEH $\pounds_\ell\, q_{ab} \triangleq 2\underleftarrow{\nabla_{(a}\ell_{b)}} \triangleq 0$. Thirdly, the volume form on Δ, is lie-dragged by and null normal in the equivalent class, $\pounds_{\xi\ell}m \triangleq 0$.
At this point one should note that the acceleration $\kappa_{(\xi\ell)}$ is in general a function on Δ. If we want NEH to obey the zeroth law of black hole mechanics, which requires constancy of the acceleration of the null normal on Δ, we should restrict it further. This is done by demanding the fourth condition in the list of boundary conditions, equation (2.2). Although this is not a single condition, (*i.e.* unlike the other three conditions, it is not guaranteed that if this condition holds for a single vector field ℓ^a, it will hold for *all* the others in the class $[\xi\ell^a]$ for any arbitrary ξ), one can always choose a class of functions ξ on Δ [44, 45], for which this reduces to a single condition. For example, if the class of function is, $\xi = F\exp(-\kappa_{(\ell)}v) + \kappa_{(\xi\ell)}/\kappa_{(\ell)}$, where $\ell^a = (\partial/\partial v)^a$ and F is a function such that $\pounds_\ell F \triangleq 0$, the condition holds for the entire equivalence class.[2]. Also note that from (2.4) that $d\omega^{(\xi\ell)}$, which is proportional to the Weyl tensor, is independent of variation of ξ. Since the Weyl tensor vanishes identically in three dimensions, we have $d\omega^{(\xi\ell)} \triangleq 0$. The equation (2.2) then gives the *zeroth law*: $d\kappa_{(\xi\ell)} \triangleq 0$.

III. WEAK ISOLATED HORIZON: DYNAMICS

In this section, we shall introduce the action for our theory and derive the laws of black hole mechanics. We shall use the first order connection formulation. This formulation is tailor-made for our set-up and the calculational simplicity will be enormous. In particular, the construction of the covariant phase-space and it's associated symplectic structure is a straightforward application of the notions used in higher dimensions [44, 45]. The use of forms also simplifies the calculation of first law and the conserved charges.
Our 3-manifold $\mathcal{M}$ will be taken to be topologically $M \times \mathbb{R}$ with boundaries. The inner null boundary will be denoted by Δ which is taken to be topologically $S^1 \times \mathbb{R}$. The initial and final space-like boundaries are denoted by M_- and M_+ respectively. The boundary at infinity will be denoted by i_0. In what follows, the inner boundary will be taken to be a WIH. In particular, this implies that the surface Δ is equipped with an equivalent class of null-normals $[\xi\ell^a]$ and follows eqn. (2.2).

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[2] This is a virtue in disguise in the sense that we can interpolate between extremal horizons, with $\kappa \triangleq 0$ to non-extremal horizons with $\kappa \neq 0$ using this ξ. In other words, we can use this formalism to accommodate extremal as non-extremal horizons in the same phase space.

A. The Action in $2+1$ dimensions

Action describing 2+1 gravity with negative cosmological constant $\Lambda = -\frac{1}{l^2}$ in first order formalism is (in our convention of $16\pi G = 1 = c$)

$$I = \int_M e^I \wedge \left(2\, dA_I + \epsilon_I{}^{JK} A_J \wedge A_K\right) + \frac{1}{3l^2}\, \epsilon^{IJK}\, e_I \wedge e_J \wedge e_K \tag{3.1}$$

where e^I are the $SO(2,1)$ orthonormal triad frame and the A^I are connections (or canonically projected local connection) of the frame-bundle with structure group $SO(2,1)$. The above action is well defined and differentiable in absence of boundaries. However, as we will show in the next subsection, with the present boundary conditions (at internal and asymptotic) at hand, the variational problem is well defined with this action itself, without any boundary term. [3]

The equations of motion are expected first order versions of the Einstein equation with cosmological constant:

$$F_I := 2dA_I + \epsilon_I{}^{JK} A_J \wedge A_K = -\frac{1}{l^2}\, \epsilon_I{}^{JK}\, e_J \wedge e_K \tag{3.2}$$

$$T_I := de_I + \epsilon_{IJK}\, e^J \wedge A^K = 0 \tag{3.3}$$

More general models for 2+1 gravity with negative cosmological constant were introduced in [29] and later studied extensively in [26, 27, 30], which without matter fields read:

$$I = a\, I_1 + b\, I_2 + \alpha_3\, I_3 + \alpha_4\, I_4 \tag{3.4}$$

where

$$\begin{aligned} I_1 &= \int_M e^I \wedge \left(2dA_I + \epsilon_I{}^{JK} A_J \wedge A_K\right) \\ I_2 &= \int_M \epsilon^{IJK} e_I \wedge e_J \wedge e_K \\ I_3 &= \int_M A^I \wedge dA_I + \frac{1}{3}\epsilon_{IJK}\, A^I \wedge A^J \wedge A^K \\ I_4 &= \int_M e^I \wedge de_I + \epsilon_{IJK}\, A^I \wedge e^J \wedge e^K \end{aligned}$$

For more references on various applications of this model and its relevance with topological massive gravity (TMG) and chiral TMG, see [6] and references therein. However one must note that this model does not reproduce the Einstein equations (3.2) and (3.3) for arbitrary values of the parameters a, b, α_3, α_4. We choose, as a special case of the above model, those values of these parameters which gives the expected equations of motions

$$a = 1 \;\; b = \frac{1}{3l^2} \;\; \alpha_3 = \frac{l}{\gamma} \;\; \alpha_4 = \frac{1}{\gamma l} \tag{3.5}$$

γ is introduced as new dimensionless parameter from 2+1 gravity perspective. Effectively (3.5) is the equation of a 3 dimensional hypersurface parametrized by G, l, γ in the 4-d parameter space of a, b, α_3, α_4.

In [3], it was established that first order 2+1 gravity can be written as Chern Simons gauge theory, the gauge group being determined by the sign of the cosmological constant. It is now explicit that 2+1 gravity does not have any local degrees of freedom, since Chern Simons theory is a topological one. Moreover, as a unique feature of 3 dimensions, all the invariances of first order gravity, ie the local Lorentz ($SO(2,1)$) transformations and arbitrarily large number of diffeomorphisms are now taken care of by finite dimensional Chern Simons gauge group, when viewed on shell. Following [3, 30], one can introduce the $SO(2,1)$ or equivalently $SL(2,\mathbb{R})$ or $SU(1,1)$ connections for a principal bundle over the same base space of the frame bundle:

$$\mathcal{A}^{(\pm)} := \left(A^I \pm \frac{e^I}{l}\right) J_I^{(\pm)}\, .$$

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[3] Strictly speaking, one should add an asymptotic boundary term to this action, which may render the whole action finite [37]. But since this has nothing to do with dynamics, *i.e.* doesn't affect the variation procedure, we omit it.

Now, it also happens that $J_I^{(\pm)}$ form two decoupled $SO(2,1)$ lie algebras:

$$\left[J_I^{(+)}, J_J^{(+)}\right] = \epsilon_{IJK}\, J^{(+)K} \quad \left[J_I^{(-)}, J_J^{(-)}\right] = \epsilon_{IJK}\, J^{(-)K}$$
$$\left[J_I^{(+)}, J_J^{(-)}\right] = 0. \tag{3.6}$$

The metric on the Lie algebra is:

$$\langle J^{(\pm)I}, J^{(\pm)J} \rangle = \frac{1}{2}\eta^{IJ}$$

It is easily verifiable that the action

$$\tilde{I} = l\left(I^{(+)} - I^{(-)}\right) \tag{3.7}$$

is same as (3.1) upto boundary terms which are guaranteed to vanish in our case. Where

$$I^{(\pm)} = \int_M \mathrm{tr}\left(\mathcal{A}^{(\pm)} \wedge d\mathcal{A}^{(\pm)} + \frac{2}{3}\mathcal{A}^{(\pm)} \wedge \mathcal{A}^{(\pm)} \wedge \mathcal{A}^{(\pm)}\right) \tag{3.8}$$

One striking feature of this formulation is that the last two terms of (3.4) can also be incorporated in terms of $\mathcal{A}^{(\pm)}$, for ($\alpha_3 = l^2\alpha_4$) as:

$$I^{(+)} + I^{(-)} = \int_M \left(A^I \wedge dA_I + \frac{1}{l^2}e^I \wedge de_I + \frac{1}{3}\epsilon_{IJK}A^I \wedge A^J \wedge A^K + \frac{1}{l^2}\epsilon_{IJK}A^I \wedge e^J \wedge e^K\right)$$

and the same equations of motion (3.2) and (3.3) are also found from varying this action. In this paper, we shall work with the following action:

$$\begin{aligned} I &= l\left(I^{(+)} - I^{(-)}\right) + \frac{l}{\gamma}\left(I^{(+)} + I^{(-)}\right) \\ &= l\left[(1/\gamma + 1)\, I^{(+)} + (1/\gamma - 1)\, I^{(-)}\right] \end{aligned} \tag{3.9}$$

with a dimensionless non-zero coupling γ. This action (3.9) upon variations with respect to $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ give equations of motion as expected from Chern-Simons theories. This imply that the connections $\mathcal{A}^{(\pm)}$ are flat:

$$\mathcal{F}_I^{(\pm)} := d\mathcal{A}_I^{(\pm)} + \epsilon_{IJK}\mathcal{A}^{(\pm)J} \wedge \mathcal{A}^{(\pm)K} = 0. \tag{3.10}$$

It is also easy to check that the above flatness conditions of these $SO(2,1)$ bundles (3.10) are equivalent to the equations of motion of general relativity (3.2), (3.3). Notice that the new action is like the Holst action [32] used in 3+1 gravity. In our case the parameter γ can be thought of being the 2+1 dimensional counterpart of the original Barbero-Immirzi parameter. Moreover the part $[I^{(+)} + I^{(-)}]$ of the action in this light qualifies to be at par with the topological (non-dynamical) term one adds with the usual Hilbert-Palatini action in $3+1$ dimensions, since this term we added (being equal to a Chern-Simons action for space-times we consider) is also non-dynamical. But more importantly the contrast is in the fact that the original action, which is dynamical in the 3+1 case is also non-dynamical here, when one considers local degrees of freedom only. However, there is a difference between the original B-I parameter and the present one. In the $3+1$ dimensions, γ parameterizes canonical transformations in the phase space of general relativity. From the canonical pair of the $SU(2)$ triad (time gauge fixed and on a spatial slice) and spin-connection one goes on finding an infinitely large set of pairs parameterized by γ. The connection is actually affected by this canonical transformation, and this whole set of parameterized connections is popularly known as the Barbero-Immirzi connection. The fact that this parameter induces canonical transformation can be checked by seeing that the symplectic structure remains invariant under the transformation on-shell. On the other hand for the case at hand, *i.e.* $2+1$ gravity, as we will see in the following sub-section that inclusion of finite γ is not a canonical transformation.
Just like in $2+1$ gravity with a negative cosmological constant, the 2 parameter family of BTZ black holes is a solution of this theory. In the standard coordinates, the solution is given by:

$$ds^2 = -N^2\, dt^2 + N^{-2}\, dr^2 + r^2(N^\phi\, dt + d\phi)^2, \tag{3.11}$$

where the lapse and the shift variables contain the two parameters M and J and are defined by:

$$N^2 = (-\frac{M}{\pi} + \frac{r^2}{l^2} + \frac{J^2}{4\pi r^2}) \quad \text{and} \quad N^\phi = -\frac{J}{2\pi r^2} \tag{3.12}$$

The horizon is defined through the zeros of the lapse function N^{ϕ} which gives the position of the horizon to be:

$$r_{\mp} = l\left[\frac{M}{2\pi}\left\{1 \mp (1-(J/Ml)^2)^{\frac{1}{2}}\right\}\right]^{\frac{1}{2}} \tag{3.13}$$

It is not difficult to see that the outer horizon (at r_+) satisfies the conditions of WIH Δ. It is a null surface with null normal $\ell^a = (\partial/\partial v)^a + N^{\phi}(r_+)\,(\partial/\partial\phi)^a$. A simple calculation also shows that $\theta_{(\ell)} \triangleq 0$. In what follows, we shall always refer back to this solution to check if our definitions for conserved charges are consistent.

B. Computing tetrads and connection on Δ

Before proceeding with the variation of the action and determining the equations of motion, it will be useful to have the values of the tetrad and connection on the null surface Δ. The usefulness of such calculation will be apparent soon. We shall assume that it is possible to fix an internal null triad (ℓ^I, n^I, m^I) such that $\ell^I n_I = -1 = -m^I m_I$ and all others zero. The internal indices will be raised and lowered with η_{IJ}. Given the internal triad basis (ℓ^I, n^I, m^I) and e^I_a, the spacetime null basis (ℓ^a, n_a, m^a) can be constructed. We shall further assume that the internal basis is annihilated by the partial derivative operator, $\partial_a\,(\ell^I, n^I, m^I) = 0$.
Using the expression of the spacetime metric in NP basis and the internal metric, we can write the tetrad e^I_a on WIH Δ as:

$$e^I_{\underleftarrow{a}} \triangleq -n_a\ell^I + m_a m^I \tag{3.14}$$

To calculate the expression of connection on Δ, we shall use the NP coefficients which can be seen in the covariant derivatives of the NP basis. They are as follows:

$$\begin{aligned}\nabla_{\underleftarrow{a}}\ell^b &\triangleq \omega^{(\ell)}_a\,\ell^b &(3.15)\\ \nabla_{\underleftarrow{a}} n_b &\triangleq -\omega^{(\ell)}_a\,n_b + U^{(\ell,m)}_a m_b &(3.16)\\ \nabla_{\underleftarrow{a}} m^b &\triangleq U^{(\ell,m)}_a\,\ell^b, &(3.17)\end{aligned}$$

where, the superscripts on the one-forms $\omega^{(\ell)}_a$ and $U^{(\ell,m)}_a$ indicate that they depend on the transformations of the corresponding basis vectors. The one-forms used in the eqn. (3.15) are compact expression of the NP coefficients. They are given by:

$$\begin{aligned}\omega^{(\ell)}_a &\triangleq (-\epsilon\,n_a + \alpha\,m_a) &(3.18)\\ U^{(\ell,m)}_a &\triangleq (-\pi\,n_a + \mu\,m_a) &(3.19)\end{aligned}$$

We will now demonstrate how the Newmann-Penrose coefficient α is fixed to be real number on Δ using topological arguments. Note from previous discussion that $d\,\omega^{(\ell)} \triangleq 0$. From the definition (6.9) we have $\underleftarrow{dm} \triangleq -\rho m \wedge n$. But because Δ is expansion-free and ℓ^a is the generator of Δ, $\rho \triangleq 0$. Hence m^a is also closed on Δ (m should not strictly be exact since $\int_{S_\Delta} m \sim$ area of horizon $\neq 0$). Since the first cohomology group of $\Delta \simeq \mathbb{R}\times S^1 \equiv \mathbb{R}$ is non-trivial, we have in general neither $\omega^{(\ell)}$ nor m_a exact. Hence there exists smooth function ς and a real number s for which

$$\omega^{(\ell)} \triangleq d\,\varsigma + s\,m \tag{3.20}$$

We now introduce a potential $\psi_{(\ell)}$ for surface gravity (or the acceleration for ℓ^a) $\kappa_{(\ell)} \triangleq \ell^a\omega^{(\ell)}_a \triangleq \epsilon$ through

$$\pounds_\ell\,\psi_{(\ell)} \triangleq \kappa_{(\ell)}.$$

Since the zeroth law implies constancy of $\kappa_{(\ell)}$ on Δ, $\psi_{(\ell)}$ can only be function of v (could be treated as the affine parameter on Δ) only. Hence $\pounds_m\psi_{(\ell)} \triangleq 0$, which implies on the other hand $d\,\psi_{(\ell)} \triangleq -\epsilon\,n$ and $\omega \triangleq d\,\psi_{(\ell)} + \alpha\,m$. It is tempting to choose $\varsigma = \psi_{(\ell)}$ by compared with (3.20). That could only be supported if Δ is axisymmetric. (Because even after choosing a triad set for which $\underleftarrow{d}\,n \triangleq 0$, we end up with $\underleftarrow{d\alpha} \wedge m \triangleq 0$, which renders $\underleftarrow{d\alpha} \triangleq 0$ only if α is axisymmetric). For that case, we conclude $\omega^{(\ell)} \triangleq (\,d\,\psi^{(\ell)} + \alpha\,m\,)$, $\alpha \in \mathbb{R}$.
Now, to calculate the connection, we use two facts. First is that the tetrad is annihilated by the covariant derivative, $\nabla_a\,e^I_b = 0$ and, secondly that partial derivative annihilates the NP internal basis so that

$$\nabla_{\underleftarrow{a}}\ell^I \triangleq A^{IJ}_a\,\ell_J. \tag{3.21}$$

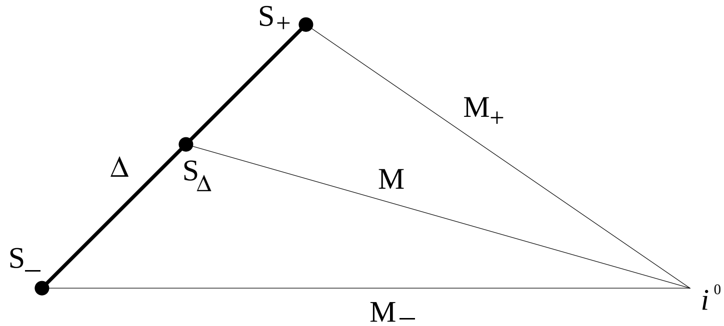

FIG. 1: Description of spacetime used in the paper. The spacetime is bounded by 2-dimensional surfaces $\Delta, M_{\mp}$ and the infinity. The horizon Δ is a 2-dimensional null surface and $M_{\mp}$ are initial and final hypersurfaces. The infinity is AdS if we work with a spacetime with negative cosmological constant.

Using equations (3.18) and (3.21) and $\epsilon_{IJK} = 3!\,\ell_{[I}n_{J}m_{K]}$, we get the following expression for pulled-back connection on Δ:

$$A^{I}_{\underleftarrow{a}} \triangleq -U^{(\ell,m)}_{a}\,\ell^{I} + \omega^{(\ell)}_{a}\,m^{I}. \tag{3.22}$$

The equation (3.22) will be used frequently in what follows.

C. Variation of the action

In the subsection (III A), we demanded that we get the first order Einstein equations of motion even by varying the generalized action. For spacetime manifolds without boundary, this is trivial to check. The task is now to vary the action to obtain the equations of motion and also to verify that the action principle is obeyed in presence of the boundaries. The variation will be over configurations which satisfy some conditions at infinity and at the inner boundary (see fig. (1)). At infinity, they satisfy some asymptotic conditions which are collected in the appendix of [41]. On the inner boundary Δ, they are subjected to the following conditions: (a) the tetrad (e) are such that the vector field $\ell^{a} = e^{a}_{I}\ell^{I}$ belongs to the equivalence class $[\xi\ell^{a}]$ and (b) Δ is a WIH. On variation, we shall get equations of motion and some surface terms. The surface terms at infinity vanish because of the asymptotic conditions whereas, as we shall show, those at WIH also vanish because of WIH boundary conditions.

Variation of the action with respect to the tetrad (e) and connection (A) leads to (for $\gamma^{2} \neq 1$):

$$de^{I} + \epsilon^{I}{}_{JK}\,e^{J} \wedge A^{K} \;=\; 0 \tag{3.23}$$

$$\text{and} \qquad dA_{I} + \frac{1}{2}\,\epsilon_{I}{}^{JK}\,A_{J} \wedge A_{K} \;=\; -\frac{1}{2l^{2}}\,\epsilon^{IJK}\,e_{I} \wedge e_{J} \wedge e_{K}. \tag{3.24}$$

The first equation above just points out that the connection A_{I} is a spin-connection and the second equation is the Einstein equation. Let us now concentrate on the surface terms. The terms on the initial and final hyper surfaces M_{-} and M_{+} vanish because of action principle. Those at the asymptotic boundary vanish because of the fall-offs at infinity. On Δ, these are given by:

$$\delta I = -\int_{\Delta}(2\,m \wedge \delta\,\omega^{(\ell)} + \frac{l}{\gamma}\,\omega^{(\ell)}\,\wedge\,\delta\,\omega^{(\ell)} + \frac{1}{l\gamma}m\,\wedge\,\delta\,m) \tag{3.25}$$

Our strategy will be to show that the integral is constant on Δ and the integrand is a total derivative so that the integral goes on to the initial and the final boundaries where the variations are zero by assumption. This will then imply that the integral itself vanishes on Δ. Note that in the above equation, $\delta\,\omega^{(\ell)}$ refers to the variation in $\omega^{(\ell)}$ among the configurations in the equivalence class $[\xi\ell^{a}]$. The relation between these are precisely given by eqn. (2.4). Now, we consider the lie derivative of the integrands by $\xi\ell$. Since $dm \triangleq 0$, it follows that $\pounds_{\xi\ell}\,m \triangleq 0$ and $\pounds_{\xi\ell}\,\omega^{(\ell)} \triangleq d(\pounds_{\xi\ell}\,ln\xi)$. Thus, in the first term, the total contribution is on the initial and final hyper surfaces M_{-} and M_{+} where the variations vanish. Identical arguments for the second and the third integrands also show that the corresponding integral vanishes. Thus, the integral is lie dragged on Δ and since the variations are fixed on the initial and final hyper surfaces, the entire integral vanishes and the action principle remains well-defined.

D. Covariant Phase Space

Analysis of the dynamics of this theory has been considerably worked out in literature [26] in the canonical framework even in presence of asymptotic boundary. A covariant phase space [46] analysis for the same theory was presented in [6] although in absence of boundaries. As we progress, we will see how apt the covariant analysis is in understanding horizon phenomena and even the conserved charges arising from asymptotic symmetries; using the general ideas of symplectic geometry. The covariant phase space is by definition the space of classical solutions (3.23) which satisfy the boundary conditions specified in the previous subsections. In other words, the covariant phase space Γ will consist of of solutions of the field equations which satisfy the boundary conditions of WIH at Δ and have fall-off conditions compatible with asymptotic conditions. In order to equip this space with a symplectic structure [4], we find the symplectic potential from variation of the Lagrangian:

$$\delta\,\mathbb{L} = d\,\Theta(\delta) + \text{terms vanishing on shell,} \tag{3.26}$$

For the Lagrangian in hand given by (3.9), the symplectic potential is given by:

$$\Theta(\delta) = -2\,(e^I\ \wedge \delta\,A_I) - \frac{l}{\gamma}\,(A^I\ \wedge \delta\,A_I) - \frac{1}{l\gamma}\,(e^I \wedge \delta\,e_I). \tag{3.27}$$

Upon antisymmetrized second variation, it gives the symplectic current J which is a phase-space two-form. For two arbitrary vector fields δ_1 and δ_2 tangent to the space of solutions, the symplectic current for (3.27) is given by following closed two form:

$$\begin{aligned} J(\delta_1,\delta_2) &= \delta_1\,\Theta(\delta_2) - \delta_2\,\Theta(\delta_1) \\ &= -2\Big[\Big(\delta_1\,e^I \wedge \delta_2\,A_I - \delta_2\,e^I \wedge \delta_1\,A_I\Big) + \frac{l}{\gamma}\,\delta_1\,A^I \wedge \delta_2\,A_I + \frac{1}{\gamma l}\,\delta_1\,e^I \wedge \delta_2\,e_I\Big] \end{aligned} \tag{3.28}$$

Since the symplectic current is closed, $d\,J(\delta_1,\delta_2) = 0$, we define the presymplectic structure on the phase-space by:

$$\Omega\,(\delta_1,\delta_2) = \int_{M_1\cup M_2\cup\Delta\cup i_0} J\,(\delta_1,\delta_2), \tag{3.29}$$

where the terms under the integral show contributions from the various boundaries (refer to figure (1)). The surfaces M_1 and M_2 are partial Cauchy slices inside the spacetime which meet Δ in S_1 and S_2 respectively. To show that the symplectic structure is independent of the choice of Cauchy surface, we again consider the function $\psi_{(\ell)}$ such that $\pounds_\ell\,\psi_{(\ell)} = \kappa_{(\ell)}$ and $\psi_{(\ell)}$ vanishes on S^1 (where the affine parameter $v = 0$). Choosing a orientation, it is not difficult to show that $J\,(\delta_1,\delta_2) \triangleq d j\,(\delta_1,\delta_2)$ so that

$$(\int_{M_1} - \int_{M_2})\,J\,(\delta_1,\delta_2) = (\int_{S_1} - \int_{S_2})\,j\,(\delta_1,\delta_2) \tag{3.30}$$

which establishes the independence of symplectic structure on choice of Cauchy surfaces.The pre-symplectic structure on the space of solutions of the theory in presence of Δ turns out to be

$$\begin{aligned} \Omega\,(\delta_1,\delta_2) &= -2\int_M \Big[\Big(\delta_1 e^I \wedge \delta_2 A_I - \delta_2 e^I \wedge \delta_1 A_I\Big) + \frac{l}{\gamma}\,\delta_1 A^I \wedge \delta_2 A_I + \frac{1}{\gamma l}\,\delta_1 e^I \wedge \delta_2 e_I\Big] \\ &-\ 2\int_{S^1}\Big(\delta_1\psi_{(\ell)}\,\delta_2\Big[\Big(\frac{l\alpha}{\gamma}+1\Big)m\Big] - \delta_2\psi_{(\ell)}\,\delta_1\Big[\Big(\frac{l\alpha}{\gamma}+1\Big)m\Big]\Big) \end{aligned} \tag{3.31}$$

We shall use (3.31) to define conserved quantities like the angular momentum and prove the first law in the next two subsections. We shall also construct the algebra of conserved charges using this symplectic structure and obtain the entropy for black holes in this theory.

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[4] To be more precise, here we will be dealing with the pre-symplectic structure, since the theory has gauge redundancy, which appear as 'degenerate directions' for the symplectic 2-form

E. Angular Momentum

We shall first introduce the concept of angular momentum starting from the symplectic structure, equation (3.31). Let us consider a fixed vector field φ^a on Δ and all those spacetimes which will have φ^a as the rotational Killing vector field on Δ. The field φ^a is assumed to satisfy certain properties. First, it should lie drag all fields in the equivalence class $[\xi\ell^a]$ ans secondly, it has closed orbits and affine parameter $\in [0, 2\pi)$. To be more precise, we can construct a submanifold Γ_φ of the covariant phase space Γ the points of which are solutions of field equations which admit a WIH $(\Delta, [\xi\ell^a], \varphi^a)$ with a rotational Killing vector field φ^a such that $\pounds_\varphi q_{ab} \triangleq 0$, $\pounds_\varphi \omega^{(l)} \triangleq 0$. Now, let us choose a vector field ϕ in $\mathcal{M}$ for each point in Γ_φ such that it matches with φ^a on Δ.
We shall now look for phase space realization of diffeomorphisms generated by this vector field ϕ^a on spacetime. Corresponding to the diffeomorphisms on spacetime, we can associate a motion in the phase space Γ_φ which is generated by the vector field $\delta_\varphi = \pounds_\varphi$. It is expected that the vector field δ_ϕ will be Hamiltonian (*i.e.* generate canonical transformations). In that case, the Hamiltonian charge for the corresponding to the rotational Killing vector field can be called the angular momentum [5]. In short, this implies that $\Omega(\delta, \delta_\phi) = \delta J^{(\phi)}$ and the angular momentum is $J^{(\phi)}$ is given by:

$$\begin{aligned} J^{(\phi)} &= -\oint_{S_\Delta} \left[(\varphi\cdot\omega)m + \frac{l}{2\gamma}(\varphi\cdot\omega)\omega + \frac{1}{2\gamma l}(\varphi\cdot m)m \right] + \oint_{S_\infty} \left[(\phi\cdot A^I)e_I + \frac{l}{2\gamma}(\phi\cdot A^I)A_I + \frac{1}{2l\gamma}(\phi\cdot e^I)e_I \right] \\ &= -J_\Delta + J_\infty \end{aligned} \tag{3.32}$$

It is then natural to interpret J_Δ to be the angular momentum on Δ. It is simple to check that for BTZ space-time the expressions for J_Δ and J_∞. It follows that $J_\Delta = (J - Ml/\gamma) = J_\infty$, leaving $J^{(\phi)} = 0$ (Note that for $\gamma \to \infty$, we get the value of angular momentum of BTZ black hole for GR in $2+1$ dimensions with a negative cosmological constant). That $J_\Delta = J_\infty$ is also supported by the fact that ϕ^a is global Killing vector in BTZ solution. However, if there are electromagnetic fields, the result differs. The value of the angular momentum at infinity J_∞ also gets contribution from the electromagnetic fields and $J^{(\phi)} \neq 0$ [41].

F. First Law

First law is associated with energy which implies that we should look first for a timelike Killing vector field on spacetime. Let us consider a time-like vector field t^a in $\mathcal{M}$ associated to each point of the phase space (live) which gives the asymptotic time translation symmetry at infinity and becomes $t^a \triangleq \xi\,\ell^a - \Omega_{(t)}\phi^a x$ on Δ, where $\Omega_{(t)}$ is a constant on Δ but may well vary on the space of histories. Just like in the previous subsection, we ask if the associated vector field δ_t on the phase-space Γ_ϕ is a Hamiltonian vector field. The associated function shall be related to the energy. In checking so, we have:

$$\Omega\,(\delta, \delta_t) = X^{(t)}(\delta)\,,$$

where,

$$X^{(t)}(\delta) = -2\kappa_{(t)}\,\delta\left((1 + \frac{l\,\alpha}{\gamma})\,a_\Delta\right) - 2\Omega_{(t)}\,\delta J_\Delta + X^{(t)}_\infty(\delta) \tag{3.33}$$

and $\kappa_{(t)}$ actually the surface gravity associated with the vector field $\xi\ell^a$. $X^{(t)}_\infty(\delta)$ involves integrals of fields at asymptotic infinity and can be evaluated using asymptotic conditions on the BTZ solution for example. A simple calculation gives:

$$X^{(t)}_\infty(\delta) = \delta\left(M - \frac{J}{\gamma l}\right)$$

Now the evolution along t^a is Hamiltonian only if right hand side of (3.33) is exact on phase space. This implies if the surface gravity is a function of area only and $\Omega_{(t)}$ a function of angular momentum only, there exists a

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[5] Since the theory we started with is background independent (has manifest diffeomorphism invariance in bulk) it is natural to expect that Hamiltonians generated by space time diffeomorphisms must consist of boundary terms, if any.

phase space function E^t_Δ such that the *first law* appears:

$$\begin{aligned} \delta E^t_\Delta &= \left[\kappa_{(\xi\ell)}\,\delta\left((1+l\alpha/\gamma)\,a_\Delta\right) + \Omega_{(t)}\,\delta J_\Delta\right] \\ &= \left(\kappa_{(\xi\ell)}\,\delta\tilde{a}_\Delta + \Omega_{(t)}\,\delta J_\Delta\right) \end{aligned} \tag{3.34}$$

where $\tilde{a}_\Delta = \left(1+\dfrac{l\alpha}{\gamma}\right)a_\Delta$. The presence of $\kappa_{(\xi\ell)}$ in the first law indicates that the first law is same for both extremal and non-extremal black holes. A mere choice of the function ξ can help us interpolate between these class of solutions. We note here that modification in the symplectic structure of the theory leaves its footprint through γ in the first law of (weak) isolated horizon mechanics. The term that plays the role of the 'area' term as it appears in this first law differs from the standard geometrical area of the horizon. If we restrict ourselves to the class of BTZ horizons, we have, [6]

$$\tilde{a}_\Delta = 2\pi\left(r_+ - r_-/\gamma\right) = a_\Delta - \frac{l\pi J}{\gamma a_\Delta} \tag{3.35}$$

G. Admissible Vector Fields and Horizon Mass

In the previous discussion we used the Hamiltonian evolution of the live time vector field t^a to deduce the first law. It is necessary and sufficient for the existence of the Hamiltonian function E^t_Δ as in (3.34) that the functions $\kappa_{(t)}, \Omega_{(t)}$ should be functions of the independent horizon parameters $\tilde{a}_\Delta$ and J_Δ only and following exactness condition should hold:

$$\frac{\partial\kappa_{(t)}}{\partial J_\Delta} = \frac{\partial\Omega_{(t)}}{\partial\tilde{a}_\Delta}. \tag{3.36}$$

However, given any vector field, it is not guaranteed that these will be satisfied. In other words, not all vector fields are Hamiltonian. Vector fields t^a for which these conditions are satisfied are admissible and there are infinite of them. We wish to find the class of admissible t^a s by solving (3.36). The essential point is to show the existence of a canonical live vector field. The horizon energy defined by this canonical live vector field is called the horizon mass. In order to proceed, we make the following change of variables for convenience:

$$(\tilde{a}_\Delta, J_\Delta) \rightarrow (R_+, R_-)$$

with

$$\begin{aligned} R_+ &= \sqrt{\frac{\gamma l}{2\pi\,(\gamma^2-1)}\left(J_\Delta + \frac{\gamma\,\tilde{a}^2_\Delta}{8\pi l}\right)} \\ R_- &= \gamma\sqrt{\frac{\gamma l}{2\pi\,(\gamma^2-1)}\left(J_\Delta + \frac{\gamma\,\tilde{a}^2_\Delta}{8\pi l}\right)} - \frac{\gamma\tilde{a}_\Delta}{4\pi} \end{aligned} \tag{3.37}$$

Now for $\kappa_{(t)}$ we wish to start with a sufficiently smooth function κ_0 of the horizon parameters. In general $\kappa_{(\ell)} \neq \kappa_0$. But we can always find a phase-function ξ in $t^a \triangleq \xi\ell^a - \Omega_{(t)}\varphi$ such that $\kappa_{(\xi\ell)} = \kappa_0$. Again, there is a canonical choice, supplied by the known solution, the BTZ one, in which there is a unique BTZ black-hole for each choice of the horizon parameters. We therefore set $\kappa_0 = \kappa_{(t)}(\text{BTZ})$, where t^a is the global time translation Killing field of the BTZ space time, and express it in terms of the newly introduced coordinates:

$$\kappa_0 = \frac{R_+^2 - R_-^2}{R_+ l^2}$$

[6] In our conventions, the double roots r_+, r_- of the BTZ lapse polynomial are related with BTZ ($\gamma \mapsto \infty$) mass (M) and angular momentum (J) as

$$M = 2\pi\,\frac{r_+^2 + r_-^2}{l^2} \quad \text{and} \quad J = 4\pi\,\frac{r_+ r_-}{l}$$

The angular velocity $\Omega_{(t)}$ satisfying (3.36) comes out as $\Omega_{(t)} = \dfrac{R_-}{lR_+}$. Using this value of angular velocity and equation (3.37)in (3.34) we have

$$\delta E^t_\Delta \;=\; \delta\left[\frac{2\pi}{l^2}\left(R_+^2 + R_-^2 - 2R_+R_-/\gamma\right)\right] \tag{3.38}$$

Now, from equations (3.37) and (3.38), we have horizon mass in terms of the independent horizon parameters:

$$M_\Delta\left(J_\Delta, \tilde{a}_\Delta\right) = \frac{\gamma J_\Delta}{l} + \frac{\gamma^2\,\tilde{a}_\Delta^2}{8\pi l^2} - \frac{\tilde{a}_\Delta}{2l^2}\sqrt{l\gamma\,\left(\gamma^2-1\right)\left(J_\Delta + \frac{\gamma\,\tilde{a}_\Delta^2}{8\pi l}\right)}.$$

It is not difficult to check that this works for BTZ black hole. Restricting to BTZ values, this reads: $M_\Delta = (M - J/\gamma l\,)$. This exactly matches with the asymptotic charge $X^{(t)}_\infty(\delta) = \delta\;(M - J/\gamma l)$ associated with asymptotic time translation vector t^a of BTZ space time as would have been expected. We must also note that the deformations of the conserved charges : angular momentum and mass under the influence of the parameter γ are exactly same as those stated in [26–28] and at the 'chiral point' ($\gamma = 1$) angular momentum and the mass become proportional to each other with opposite sign.

IV. COVARIANT PHASE SPACE REALIZATION OF ASYMPTOTIC SYMMETRY ALGEBRA

It has been suggested that microscopic details which explain the thermodynamics of black holes is independent of any theory of quantum gravity. If this is taken seriously, it implies that the microstates that describe black hole spacetime can be understood from a principle which is expected to govern all quantum gravity theory. It then seems natural to use the arguments of symmetry. Whatever be the theory of quantum gravity, it must at least preserve a part of the symmetries of classical theory. Study of asymptotic symmetries have been advocated to serve this purpose and has achieved striking success in reproducing the Bekenstein-Hawking formula. This issue was first addressed in the context of $2+1$ gravity (with negative cosmological constant) by [20].
In this issue we note that diffeomorphisms which are gauges for any theory of gravity become physical symmetry at the boundaries of the space time manifold by physical requirements (boundary conditions). For example, in $3+1$ dimensional asymptotically flat space times one naturally identifies a time like vector field at asymptotic infinity as the unique time translation (Killing) as in Minkowski space time and fixes it once and for all. This fixes the diffeomorphisms partially and play the role of a physical symmetry. Only then we can associate a Hamiltonian or Noether charge with time which is the ADM mass. In [21], the authors considered diffeomorphisms generated by asymptotic vector fields which are a bit 'relaxed Killing symmetries' of the asymptotic metric in a $2+1$ dimensional space time and showed that they form the pair of affine Witt algebra (2D conformal algebra, or deformation algebra of S^1) as opposed to $SO(2,2)$, the isometry group of AdS_3. We will show that those vector fields actually generate flows in the phase space which are at least locally Hamiltonian and find the corresponding Hamiltonians (hence qualifying as physical symmetries), *i.e.* charges in the covariant phase space framework. The preference for this frame work is firstly due to its manifest covariant nature and secondly for its immense calculational simplicity, as compared to canonical framework [18].
According to the suggestion mentioned above, this immediately implies that the quantum theory describing the microstates of black holes is a conformal field theory. The simple use of central charges in the Cardy formula determines the asymptotic density of quantum states of black holes which have same mass and angular momentum and approach the asymptotic configuration of a classical BTZ black hole; and eventually the Bekenstein-Hawking result. We shall use the covariant phase-space formulation to compute black hole entropy in this theory.
Let us gather the essential details for the asymptotic analysis. For the BTZ solution (3.11), the tetrads and connections are given by:

$$e^0 = N\,dt,\;\; e^1 = N^{-1}\,dr \;\text{ and }\; e^2 = r\,(\,d\phi + N_\phi\,dt\,)$$

and

$$A^0 = -N\,d\phi,\;\; A^1 = N^{-1}\,N_\phi\,dr \;\text{ and }\; A^2 = -\frac{r}{l^2}\,dt - r\,N_\phi\,d\phi,$$

where $N^2 = \left[\dfrac{r^2}{l^2} - \dfrac{M}{\pi} + \dfrac{J^2}{4\pi^2 r^2}\right]$, $N_\phi = \left[\dfrac{J}{2\pi r^2}\right]$ and the internal metric is $\eta_{IJ} = \text{diag}(+,-,-)$.
The asymptotic form of these variables match with the AdS ones as expected upto different orders of $1/r$ [26, 28].

The asymptotic vector fields which generate diffeomorphisms preserving the asymptotic AdS structure (much milder than the BTZ solution) are given by:

$$\xi_n := \exp\left(inx_+\right)\left[l\left(1-\frac{l^2n^2}{2r^2}\right)\partial_t - inr\,\partial_r + \left(1+\frac{l^2n^2}{2r^2}\right)\partial_\phi\right]$$

with n an integer and $x_+ = (t/l+\phi)$. It is easy to check that the vector fields satisfy the affine Witt algebra:

$$[\xi_n, \xi_m] = -i(n-m)\,\xi_{n+m} \tag{4.1}$$

We now want to investigate if the algebra of the vector fields on the space-time manifold is also realised on the phase space *i.e* the Hamiltonian functions (or the generators of diffeomorphisms) corresponding to the vector fields ξ_n^a also satisfy the affine algebra. To see this, we first associate a phase space vector field δ_{ξ_n} to each element ξ_n of the algebra such that δ_{ξ_n} acts as $\pounds_{\xi_n}$ on dynamical variables[7]. Secondly, we need the symplectic structure which will enable us to construct the Hamiltonian functions as has been described in the previous sections (see (III E) and (III F)). Since we are interested in the asymptotic analysis, we will be interested in the contribution to the symplectic structure from the asymptopia or S_∞. If an internal boundary like NEH is present we can assume that the vector fields whose asymptotic forms are as ξ_n^a above vanish on that boundary. From this point of view, for any arbitrary vector field ξ_n^a which vanish on any internal boundary (in this section, we shall reinstate $16\pi G$ but shall choose $c=h=1$):

$$8\pi G\,\Omega(\delta,\delta_\xi) = \oint_{S_\infty}\left[\left(\xi\cdot e^I\right)\delta \underrightarrow{A_I} + \left(\xi\cdot A^I\right)\delta\underrightarrow{e_I} + \frac{l}{\gamma}\left(\xi\cdot A^I\right)\delta\underrightarrow{A_I} + \frac{1}{l\gamma}\left(\xi\cdot e^I\right)\delta\underrightarrow{e_I}\right] \tag{4.2}$$

The under right arrows indicate pull-back of the forms on S_∞. Therefore the second and the fourth term in the integral do not contribute. Only the internal component e_2 (as given above) survives under the pull back which is given by $-rd\phi$. This being a phase space constant, the action of δ on it vanishes. Hence, we get

$$8\pi G\,\Omega(\delta,\delta_\xi) = \oint_{S_\infty}\left[\xi\cdot(e^I+\frac{l}{\gamma}A^I)\right]\delta\underrightarrow{A_I} \tag{4.3}$$

for any arbitrary vector field ξ. Using the above expressions of the fields asymptotically, we have

$$8\pi G\,\Omega(\delta,\delta_{\xi_n}) = \left(1-\frac{1}{\gamma}\right)\delta\,(\,lM+J\,)\;\delta_{n,0}$$

hence δ_{ξ_n} are at least locally hamiltonian for all n. We also note using (4.1) that $\delta_{[\xi_n,\xi_m]}$ is also a Hamiltonian vector field with $\delta H[\{\xi_n,\xi_m\}]$ given by the right hand side of the following equation

$$8\pi G\,\Omega(\delta,\delta_{[\xi_n,\xi_m]}) = -i(n-m)\left(1-\frac{1}{\gamma}\right)\delta\,(\,lM+J\,)\;\delta_{m+n,0} \tag{4.4}$$

We shall now determine the current algebra of the Hamiltonian functions (*i.e.* $\{H_{\xi_n}, H_{\xi_m}\}$) generated by the Hamiltonian vector fields δ_{ξ_n} and δ_{ξ_m} for arbitrary n, m. This will be given by:

$$8\pi G\,\Omega(\delta_{\xi_m},\delta_{\xi_n}) = \oint_{S_\infty}\left[\xi_n\cdot(e^I+\frac{l}{\gamma}A^I)\right]\delta_{\xi_m}\underrightarrow{A_I} \tag{4.5}$$

It is now important that we first pull back A_I and then calculate the action of δ_{ξ_m} on it as Lie derivative. After some lines of calculation, we find:

$$\begin{aligned} 8\pi G\,\Omega(\delta_{\xi_m},\delta_{\xi_n}) &= -2in\left(1-\frac{1}{\gamma}\right)(\,J+lM\,)\,\delta_{m+n,0} + il\pi n^3\left(1-\frac{1}{\gamma}\right)\delta_{m+n,0} + O(\frac{1}{r^2}) \\ &= -i(n-m)\left(1-\frac{1}{\gamma}\right)(\,J+lM\,)\,\delta_{m+n,0} + il\pi n^3\left(1-\frac{1}{\gamma}\right)\delta_{m+n,0} \end{aligned} \tag{4.6}$$

[7] This is because vector fields on the space time manifold work as generators of infinitesimal diffeomorphisms

Comparing (4.4) and (4.6) we infer that the asymptotic diffeomorphism algebra (4.1) is exactly realized at the canonical level (as a current algebra) except a 'central term' $-il\pi n^3\left(1-\frac{1}{\gamma}\right)\delta_{m+n,0}$. This is not surprising, although all the vector fields δ_{ξ_n} were Hamiltonian. The second cohomology group of the Witt algebra [8] is not trivial. A theorem of symplectic geometry states that in this case the action of the algebra is not Hamiltonian and moment maps donot exist, which on the other hand implies that the action of the lie algebra on phase space is not hamiltonian [47] [9], *i.e.*

$$\delta\,\Omega\,(\,\delta_{\xi_m}\,,\,\delta_{\xi_n}\,) \neq \Omega\,(\,\delta_{\,[\xi_m,\xi_n]}\,,\,\delta\,). \tag{4.7}$$

All of this calculation was done choosing the right moving vector fields. There also are a set of left moving vector fields which preserve the asymptotic structure:

$$\tilde{\xi}_n := \exp(inx_-)\left[l\left(1-\frac{l^2n^2}{2r^2}\right)\partial_t - inr\,\partial_r - \left(1+\frac{l^2n^2}{2r^2}\right)\partial_\phi\right]$$

where $x_- = (t/l - \phi)$ Proceeding along the very same route as before, we again end up with the result that canonical realization of this asymptotic symmetries are also realized exactly upto a central term, which now becomes $= -il\pi n^3\left(1+\frac{1}{\gamma}\right)\delta_{m+n,0}$
¿From the definition of the central charge of Virasoro algebra, which is the centrally extended version of the Witt algebra, we arrive at the exact formulas for the central charges for the right and left moving algebras respectively :

$$c_\pm = \frac{3l}{2G}\left(1 \pm \frac{1}{\gamma}\right)$$

Once we have the central charges, we can apply the Cardy formula to the BTZ solution to obtain the black hole entropy:

$$\begin{aligned} S &= \frac{2\pi\,r_+}{4G} - \frac{2\pi\,r_-}{4G\gamma} = \left(a_\Delta - \frac{l\pi J}{\gamma a_\Delta}\right)/4G \\ &= \frac{\tilde{a}}{4G} \end{aligned} \tag{4.8}$$

where r_+ and r_- are the radii of the outer and inner horizon, respectively. If we consider the thermodynamic analogy of the first law of black hole mechanics (3.34) (derived for general spacetimes only requiring presence of a weakly isolated horizon only from classical symplectic geometric considerations), we observe that $S \sim \tilde{a}$. Curiously, even in the quantum result (4.8), the entropy-modified area relation continues to hold.

V. CONCLUSION

Let us recollect the main findings of this paper. Firstly, we introduced the concept of WIH in 2 + 1 dimensions. The boundary conditions which have been imposed on a 2-dimensional null surface are much weaker than the ones suggested in [41]. Our boundary conditions are satisfied by a equivalance class of null normals which are related by functions, $[\xi\ell^a]$ rather than constants, $[c\ell^a]$ as was first proposed in [41]. The advantage of such generalisation lies in the fact that it becomes possible to include extremal as well as non-extremal solutions in the same space of solutions. Just by choosing the function ξ, one can move from a non-zero $\kappa_{(\ell)}$ to a vanishing $\kappa_{(\xi\ell)}$ (see equation (2.1)) which essentially is like taking extremal limits in phase-space. We also established that the zeroth law (for all solutions in this extended space of solutions) follows quite trivially from the boundary conditions.

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[8] For any real lie algebra $\mathcal{G}$ and its dual $\mathcal{G}^*$ a skew symmetric bilinear map $\alpha \in \mathcal{G}^* \wedge \mathcal{G}^*$ is said to be a *cocycle* if $\alpha\,([A,B]\,,C) + \alpha\,([B,C]\,,A) + \alpha\,([C,A]\,,B) = 0$ for all $A,B,C \in \mathcal{G}$ and $[,]$ is the usual product on $\mathcal{G}$. The elements $\eth f$ $(f \in \mathcal{G}^*)$ defined via $\eth f(A,B) = \frac{1}{2}f([A,B])$, automatically cocycles by Jacobi identity, are called *coboundary*. Let us define an equivalence $\sim$ as: two cocycles $\alpha \sim \beta$ if $\alpha = \beta + \eth g$ for any $g \in \mathcal{G}^*$. Now one defines $H^2\mathcal{G}$ as the additive group of equivalence classes found through the modulo action of the equivalence relation. All semi simple lie algebras have trivial second cohomology.

[9] If $J[\xi_m]$ and $J[\xi_n]$ are Hamiltonians (calculated in the canonical phase-space) corresponding to the vector fields ξ_m and ξ_n,then $J[[\{\xi_m,\xi_n\}] \neq \{J[\xi_m], J[\xi_n]\}$ where $\{J[\xi_m], J[\xi_n]\} =: \delta_{\xi_n}\,J[\xi_m] = -\delta_{\xi_n}\,J[\xi_m]$.

Secondly, we have explicitly shown that in presence of an internal boundary satisfying the boundary conditions of a WIH, the variational principle for the generalised $2 + 1$ dimensional theory remains well-defined. This enable us to take the third step where we have constructed the covariant phase-space of this theory. The covariant phase-space now contains all solutions of the γ-dependent theory which satisfy the WIH boundary conditions at infinity. As expected, extremal as well as non-extremal solutions form a part of this phase-space. We then went on to define the angular momentum as a Hamiltonian function corresponding to the rotational Killing vector field on the horizon. It was also explicitly shown that for the BTZ solution, the angular momentum defined in this manner matches with the expected result.
Thirdly, we established the first law of black hole mechanics directly from the covariant phase-space, for isolated horizons. Instead of the usual horizon area term one encounters in this law, we find a modification due to the γ factor. This is a completely new result in this family of theories. It arose that the first law is the necessarry and sufficient condition for existence of a timelike Hamiltonian vector field on the covariant phase-space. However, not all timelike vector fields are Hamiltonian on phase-space, there exists some which are admissible (there are in fact infinite of them). The canonical choice for these admissible vector fields are constructed too. Quite interestingly, the first law for the WIH formulation, equation (3.34), contains $\kappa_{(\xi\ell)}$. This implies that the first law holds for all solutions, extremal as well as non-extremal. However, the thermodynamic implications of the first law can only be extracted for non-extremal solutions since for the extremal ones, the first law is trivial. However, we expect that since all solutions are equivalent from the point of view of WIH bounhdary conditions, the entropy of both class of black hole solutions will be same.
Using asymptotic analysis, we have calculated the entropy of black holes for the theory under consideration. Contary to the usual approach, we construct the algebra of diffeomorphism generating Hamiltonian functions directly from the covariant phase-space. As usual, we see that the algebra does not match with the Hamiltonian function for the commutator of the asymptotic vector fields. The difference is the central extension. In other words, the algebra of spacetime vector fields is not realised on the covariant phase-space. The Cardy formula then gives the entropy directly which matches with the one expected from the first law. The entropy however not only depends on the geometrical area but also on of other quantities like the parameter of the solution J and the γ-parameter of the theory (equation (4.8)). Keeping the thermodynamic analogy of laws of black hole mechanics in mind and concentrating on the BTZ black hole, one observes that there is a perfect harmony between this result and the modified first law. Also recall that our methods do not rely on existence of bifurcation spheres and applies equally to extremal and non-extremal black holes. To our knowledge, this has not been reproduced earlier since the phase space of Killing horizons which satisfy laws of mechanics do not contain extremal solutions.
Our analysis for the computation of entropy is based on asymptotic symmetry analysis. The principle of using symmetry arguments to determine the density of states for black hole is attractive, it does not depend on the details of quantum gravity. The asymptotic analysis has a major drawback- it seems to be equally applicable for any massive object placed in place of a black hole. Since such objects are not known to behave like black holes, it is not clear where to attribute such large number of density of states. One must directly look at the near-horizon symmetry vector fields for further understanding [48]. However, a more interesting step would be to determine the horizon microstates as is done in $3 + 1$ dimensions. In this case, it arises from classical considerations that the degrees of freedom that reside on a WIH in $3 + 1$ dimensions is a Chern-Simons theory. Quantization of this theory gives an estimate of the states that contribute to a fixed area horizon and the entropy turns out to be proportional to area. This has not been reproduced in $2 + 1$ dimensions still and will be investigated in future in order to compliment these new findings already present in this paper.

VI. APPENDIX

The Newman-Penrose formalism for $2 + 1$ dimensions

In order to make the article self-contained we summarise here the analogue of Newman-Penrose formalism in 2+1 dimensions, which was in detail described in [41]. We will use a triad consisting of two null vectors ℓ^a and

n^a and a *real*[10] space-like vector m^a, subject to:

$$\ell\cdot\ell = n\cdot n = 0,\ \ m\cdot m = 1 \tag{6.1}$$
$$\ell\cdot m = n\cdot m = 0 \tag{6.2}$$
$$\ell\cdot n = -1. \tag{6.3}$$

The space-time metric g_{ab} can be expressed as

$$g_{ab} = -2\,\ell_{(a}n_{b)} + m_a m_b\,, \tag{6.4}$$

and its inverse g^{ab} is defined to satisfy

$$g^{ab} = -2\,\ell^{(a}n^{b)} + m^a m^b. \tag{6.5}$$

It is then easy to verify that the expression for the triad is just

$$e^I_a = -\ell_a n^I - n_a \ell^I + m_a m^I. \tag{6.6}$$

Just as in the 3 + 1 case, we express the connection in the chosen triad basis, the connection coefficients being the new N-P coefficients (the γ defined below is not to be confused with the Barbero-Immirzi parameter):

$$\begin{aligned}\nabla_a\ell_b &= -\epsilon\, n_a\ell_b + \kappa_{\rm NP}\, n_a m_b - \gamma\,\ell_a\ell_b\\ &\quad +\tau\,\ell_a m_b + \alpha\, m_a\ell_b - \rho m_a m_b \qquad (6.7)\\ \nabla_a n_b &= \epsilon\, n_a n_b - \pi\, n_a m_b + \gamma\,\ell_a n_b\\ &\quad -\nu\,\ell_a m_b - \alpha\, m_a n_b + \mu\, m_a m_b \qquad (6.8)\\ \nabla_a m_b &= \kappa_{\rm NP}\, n_a n_b - \pi\, n_a\ell_b + \tau\,\ell_a n_b\\ &\quad -\nu\,\ell_a\ell_b - \rho\, m_a n_b + \mu\, m_a\ell_b \qquad (6.9)\end{aligned}$$

It then simply follows from the expressions above that $\nabla_a\,\ell^a = (\epsilon-\rho), \nabla_a\, n^a = (\mu-\gamma)and\nabla_a\, m^a = (\pi-\tau)$. Now we wish to expand the connection 1-form A^I_a in the triad basis with N-P coefficients slated above as coefficients. In order to do so we note that for an arbitrary 1-form v_a which may be mapped uniquely to an $SO(2,1)$ frame element $v_I = v_a e^a_I$. Then, for $\nabla_a v_b = A^I_{aJ}\, v^J\, e_{Ib}$, and using $A_{a\ I}{}^J = \epsilon_{KI}{}^J A^K_a$, we arrive at the expression:

$$\begin{aligned}A^K_a &= (\pi n_a + \nu\ell_a - \mu m_a)\ell^K + (\kappa_{\rm NP}\, n_a + \tau\ell_a - \rho m_a)\, n^K\\ &\quad +(-\epsilon n_a - \gamma\ell_a + \alpha m_a)\, m^K \qquad (6.10)\end{aligned}$$

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